Estimation of McKenzie sums Section 5

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Abstract

Quick write up of the sums in question.

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divisor function, fractional part

Results

Let τ be the divisor function whose companion Dirichlet series is $\zeta^2(s) = \sum_{n=1}^\infty \tau(n)n^{-s}$, Re $s > 1$. I.e., τ is the number of (positive) divisors of n, including 1 and n. We consider the fractional part multiple sums ν_k of Section 5 of [McKenzie]. We have

Proposition. We have

$$\nu_2(n) = (1-\gamma)n + O(\sqrt{n}). \tag{1}$$

More generally, we have

$$\nu_k(n) = \frac{(1-\gamma)}{(k-2)!} n \ln^{k-2} n + O(n \ln^{k-3} n). \tag{2}$$

We will repeatedly use for the harmonic number $H_n \equiv \sum_{k=1}^n 1/k = \ln n + \gamma + 1/2n + O(1/n^2)$, where γ is the Euler constant, without further comment. We let $S_\tau(n) = \sum_{j \leq n} \tau(j)$ and have the identity

$$S_\tau(n) = \sum_{j \leq n} \left[\frac{n}{j}\right]. \tag{3}$$

Using $[x] = x - \{x\}$, where $\{x\}$ denotes the fractional part of x, we quickly arrive at

$$S_\tau(n) = n \ln n + \gamma n - \sum_{j \leq n} \left\{\frac{n}{j}\right\} + O(1). \tag{4}$$

Together with the known result

$$S_\tau(n) = n \ln n + (2\gamma - 1)n + O(\sqrt{n}), \tag{5}$$

we obtain

$$\sum_{j \leq n} \left\{\frac{n}{j}\right\} = (1-\gamma)n + O(\sqrt{n}). \tag{6}$$

The latter sum is precisely $\nu_2(n)$, and we have verified the conjectured order.

Remark. The errror term in (5) can be improved. As well, its optimal order is presumably $O(n^{1/4})$.

For $\nu_3(n)$ we use the result for $\nu_2(n)$. We have

$$\nu_3(n) = \sum_{j=2}^{n} \sum_{k=2}^{n/j} \left\{ \frac{n}{jk} \right\}$$

$$= \sum_{j=2}^{n} \left[(1-\gamma)\frac{n}{j} + O(\sqrt{n/j}) \right]$$

$$= (1-\gamma)n[\ln n + \gamma - 1 + o(1) + O(n)]$$

$$= (1-\gamma)n \ln n + O(n). \tag{7}$$

Similarly, we find $\nu_4(n) = \frac{(1-\gamma)}{2} n \ln^2 n + O(n \ln n)$.

For now, lets also consider $\nu_5(n)$ explicitly. We have

$$\nu_5(n) = \sum_{j_1=2}^{n} \sum_{j_2=2}^{n/j_1} \sum_{j_3=2}^{n/j_1 j_2} \sum_{j_4=2}^{n/j_1 j_2 j_3} \left\{ \frac{n}{j_1 j_2 j_3 j_4} \right\}$$

$$= \sum_{j_1=2}^{n} \sum_{j_2=2}^{n/j_1} \sum_{j_3=2}^{n/j_1 j_2} \left[\frac{(1-\gamma)n}{j_1 j_2 j_3} + O\left(\sqrt{\frac{n}{j_1 j_2 j_3}}\right) \right]$$

$$= \sum_{j_1=2}^{n} \sum_{j_2=2}^{n/j_1} \left[\frac{(1-\gamma)n}{j_1 j_2} \ln\left(\frac{n}{j_1 j_2}\right) + O\left(\frac{n}{j_1 j_2}\right) \right]$$

$$= \sum_{j_1=2}^{n} \left[\frac{(1-\gamma)n}{2 j_1} \ln^2\left(\frac{n}{2 j_1}\right) + O\left(\frac{n}{j_1} \ln\left(\frac{n}{j_1}\right)\right) \right]$$

$$= \frac{(1-\gamma)}{6} n \ln^3 n + O(n \ln^2 n). \tag{8}$$

By induction, we have (2).

Or we may perform a direct computation, using successively the elementary comparison integral $\int \frac{\ln^k x}{x} dx = \frac{1}{k+1} \ln^{k+1} x$.

Remarks. The error term in (2) is a poor one.

Per (5.5) in [McKenzie], we have the leading term from (2)

$$\sum_{\ell=2}^{\infty} \frac{z^\ell}{\ell} \nu_\ell(n) = z^2 \sum_{\ell=0}^{\infty} \frac{z^\ell}{(\ell+2)!} \nu_{\ell+2}(n) = (1-\gamma) z^2 n^{z+1}. \tag{9}$$

Then we have $\pi^*(n,1) = s(n,1) + 1 - \gamma + O(?)$.

We recall that the constant

$$1 - \gamma = \int_1^\infty \frac{\{t\}}{t^2} dt. \tag{10}$$

We let $\text{Ci}(x) = -\int_x^\infty \cos v (dv/v)$ be the cosine integral. From (10) we have

Corollary. We have

$$\sum_{j=1}^{\infty} \text{Ci}(2\pi j) = \frac{1}{2}\left(\frac{1}{2} - \gamma\right). \tag{11}$$

Proof. We recall that $P_1(x) = B_1(x - [x]) = x - [x] - 1/2$, the first periodized Bernoulli polynomial, has the standard Fourier series [nbs] (p. 805),

$$P_1(x) = -\sum_{j=1}^{\infty} \frac{\sin(2\pi j x)}{\pi j}. \tag{11}$$

Inserted into (10), we have

$$\begin{aligned} \int_1^\infty \frac{\{t\}}{t^2} dt &= \frac{1}{2} - \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{1}{j} \sin(2\pi j t) \frac{dt}{t^2} \\ &= \frac{1}{2} + 2 \sum_{j=1}^{\infty} \text{Ci}(2\pi j), \end{aligned} \tag{12}$$

where we integrated by parts and made a simple change of variable. Therefore, (11) follows.

Discussion. The relation $\int_1^\infty \frac{\{t\}}{t^2}dt = \frac{1}{2} + 2\sum_{j=1}^\infty \mathrm{Ci}(2\pi j)$ readily shows how to develop $1-\gamma$ from $1/2$ with a series of corrections, and the leading terms in the corrections are easily written.

By successively integrating by parts, we easily have an asymptotic form for the cosine integral,

$$\mathrm{Ci}(x) = \frac{\sin x}{x}\left(1 - \frac{2!}{x^2} + \ldots\right) - \frac{\cos x}{x}\left(\frac{1}{x} - \frac{3!}{x^3} + \ldots\right). \tag{13}$$

Therefore, the first three correction terms $\sum_{j=1}^3 \mathrm{Ci}(2\pi j)$ add to $-1/24 + 1/240 - 1/504 = -199/5040$. The series (13) is a divergent one, but it conveniently gives the leading corrections. As to be expected, the formal series

$$\sum_{j=1}^\infty \mathrm{Ci}(2\pi j) \sim -\sum_{k=0}^\infty (-1)^k \frac{(2k+1)!}{(2\pi)^{2k+2}}\zeta(2k+2), \tag{14}$$

is divergent.